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Cubic Diophantine Equation with Three Unknowns

$$(3a+5)x^2 - 3ay^2 = 125z^3$$

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Abstract: The non-homogeneous ternary cubic Diophantine equation given by $(3a+5)x^2 - 3ay^2 = 125z^3$ is considered. Different patterns of non-zero distinct integer solutions to the above equation are obtained when a=1 and a=2. For each of there cases, a few interesting properties between the solutions and special numbers are presented.

Keywords: Non-homogeneous, ternary cubic Diophantine equation and integral solutions.

1. INTRODUCTION:

Integral solutions for the homogeneous or non-homogeneous Diophantine cubic equation is an interesting concept as it can seen from [1-3]. In [4-10] a few special cases of cubic Diophantine equation with four unknowns are studied.In [11-13] cubic equations with five unknowns are studied for their integral solutions.In[14,15] cubic equations with six unknowns are studied for their integral solutions.This communication concerns with yet another cubic Diophantine equation with three unknowns $(3a+5)x^2 - 3ay^2 = 125z^3$. A few relations among the solutions are presented.

Notations:

- $T_{m,n}$ -Polygonal number of rank n with size m.
- P_m^n - Pyramidal number of rank n with size m.
- Pr_n - Pronic number of rank n.
- $CP_{m,n}$ -Centered Pyramidal number of rank n with size m.
- Ky_n -Keynea number of rank n.
- PP_n - Pentagonal Pyramidal number of rank n.

2. METHOD OF ANALYSIS:

The Ternary cubic Diophantine equation given by

$$(3a+5)x^2 - 3ay^2 = 125z^3 \quad (1)$$

The substitution of the linear transformation

$$x=X+3aT,\ y=X+(3a+5)T \quad (2)$$

in (1) leads to $X^2-3a(3a+5)T^2=25z^3$ (3)

Assume

$$z(\alpha,\beta,a) = \alpha^2 - 3a(3a+5)\beta^2,\ \alpha,\beta > 0 \quad (4)$$

Write 25 as

$$25 = \left[(6a+5)+2\sqrt{3a(3a+5)}\right]\left[(6a+5)-2\sqrt{3a(3a+5)}\right] \quad (5)$$

Substituting (4)and (5)in (3)and employing the method of factorization, define

$$(X+\sqrt{3a(3a+5)}T)=\left[(6a+5)+2\sqrt{3a(3a+5)}\right]\left[\alpha+\sqrt{3a(3a+5)}\,T\right]^3$$

Equating the rational and irrational parts on both sides, we get

$$X=(6a+5)(\alpha^3+9a(3a+5)\alpha\beta^2)+6a(3a+5)(3\alpha^2\beta+3a(3a+5)\beta^3)$$
$$T=(6a+5)(3\alpha^2\beta+3a(3a+5)\beta^3)+2(\alpha^3+9a(3a+5)\alpha\beta^2)$$

Substituting the value of X and T in (2), we get

$$\begin{aligned}x(\alpha,\beta,a)&=(12a+5)(\alpha^3+9a(3a+5)\alpha\beta^2)+9a(4a+5)(3\alpha^2\beta+3a(3a+5)\beta^3)\\ y(\alpha,\beta,a)&=(12a+15)(\alpha^3+9a(3a+5)\alpha\beta^2)+(36a^2+75a+25)(3\alpha^2\beta+3a(3a+5)\beta^3)\end{aligned} \quad (6)$$

Thus (4) and (6) represent non-zero distinct integral solutions of (1).

Properties:

- $x(\alpha,\beta,a)-y(\alpha,\beta,a)\equiv 0 \pmod{5T}$
- $(3a+5)x^2(\alpha,\beta,a)-ay^2(\alpha,\beta,a)$ is a cubical integer
- $y(a,a,1)-x(a,a,1)\equiv 0 \pmod{5T}$
- $3z(1,2,-1)$ is a cubical integer

To analyses the nature of solution one has to go in for particular values of a, in equation (1).For the sake of simplicity and clear understanding we exhibit below the integer solution of (1) along with the properties for the cases a=1 and a=2.

Case: 1

Let a=1,

(3) becomes $X^2-24T^2=25z^3$ (7)

Assume $z(\alpha,\beta)=\alpha^2-24\beta^2,\ \alpha,\beta>0$ (8)

Illustration-1:

Write 25 as $25=(7+\sqrt{24})(7-\sqrt{24})$ (9)

Substituting (8) and (9) in (7) and employing the method of factorization, define

$$(X+\sqrt{24}T)=(7+\sqrt{24})(\alpha+\sqrt{24}\beta)^3$$

Equating the rational and irrational parts on both sides, we get

$$X = 7\alpha^3 + 72\alpha^2\beta + 504\alpha\beta^2 + 576\beta^3$$
$$T = \alpha^3 + 21\alpha^2\beta + 72\alpha\beta^2 + 168\beta^3$$

Substituting the values of X and T in (2,) we get

$$\begin{aligned} x(\alpha,\beta) &= 10\alpha^3 + 135\alpha^2\beta + 720\alpha\beta^2 + 1080\beta^3 \\ y(\alpha,\beta) &= 15\alpha^3 + 240\alpha^2\beta + 1080\alpha\beta^2 + 1920\beta^3 \end{aligned} \tag{10}$$

Thus (8)and (10) represent non-zero distinct integral solutions of (1).

Properties:

- $10y(\alpha,\alpha+1) - 15x(\alpha,\alpha+1) - 750P_\alpha^5 + 3000CP_\alpha^6 \equiv 0$
- $192x(\beta+1,\beta) - 108y(\beta+1,\beta) - 4320P_\beta^5 + 300CP_\beta^6 \equiv 0$
- $10y(2\alpha+1,1) - 15x(2\alpha+1,1) = 375Ky_\alpha + 3750$
- $z(1,1)^2$ is a 12th centered octagonal number

Illustration-2:

Write 25 as $25 = (11+2\sqrt{24})(11-2\sqrt{24})$

Replacing the above process the non-zero distinct integral solutions of (1) are

$$x(\alpha,\beta) = 17\alpha^3 + 243\alpha^2\beta + 1224\alpha\beta^2 + 1944\beta^3$$
$$y(\alpha,\beta) = 27\alpha^3 + 408\alpha^2\beta + 1944\alpha\beta^2 + 3264\beta^3$$
$$z(\alpha,\beta) = \alpha^2 - 24\beta^2$$

Properties:

- $17y(\alpha,\alpha+1) - 27x(\alpha,\alpha+1) = 750PP_\alpha - 3000CP_\alpha^6$
- $408x(1,2\beta+1) - 243y(1,2\beta+1) = 27000Ky_\beta + 54375$
- $408x(\beta+1,\beta) - 243y(\beta+1,\beta) + 375CP_\beta^6 + 54000P_\beta^5 \equiv 0$
- $17y(2\alpha+1,1) - 27x(2\alpha+1,1) = 375Ky_\alpha + 3750$

Illustration-3:

Write 25 as $25 = (25+5\sqrt{24})(25-5\sqrt{24})$

Replacing the above process as in illustration-1, the corresponding non-zero distinct integral solutions of (1) are

$$x(\alpha,\beta) = 40\alpha^3 + 585\alpha^2\beta + 2880\alpha\beta^2 + 4680\beta^3$$
$$y(\alpha,\beta) = 65\alpha^3 + 960\alpha^2\beta + 4680\alpha\beta^2 + 7680\beta^3$$
$$z(\alpha,\beta) = \alpha^2 - 24\beta^2$$

Properties:

- $40y(\alpha,1) - 65x(\alpha,1) - 375\,\mathrm{Pr}_\alpha - Ky_5 = 1913(\mathrm{mod}\,375)$
- $960x(1,2\beta+1) - 585y(1,2\beta+1) = 27000\,Ky_\beta + 54375$
- $768x(\beta+1,\beta) - 468y(\beta+1,\beta) + 300CP_\beta^6 - 4320\,PP_\beta \equiv 0$
- $x(\alpha,1) - 80PP_\alpha - 545\,\mathrm{Pr}_\alpha \equiv 4680(\mathrm{mod}\,2335)$

Case:2

Let a=2

(3) becomes $X^2 - 66T^2 = 25z^3$ (11)

Assume $z(\alpha,\beta) = \alpha^2 - 66\beta^2,\ \alpha,\beta > 0$ (12)

Illustration-1:

Write 25 as $25 = (17 + 2\sqrt{66})(17 - 2\sqrt{66})$ (13)

Substituting (12) and (13) in (11) and employing the method of factorization, define

$$(X + \sqrt{66}T) = (17 + 2\sqrt{66})(\alpha + \sqrt{66}\beta)^3$$

Equating the rational and irrational parts on both sides, we get

$$X = 17\alpha^3 + 396\alpha^2\beta + 3366\alpha\beta^2 + 8712\beta^3$$
$$T = 2\alpha^3 + 51\alpha^2\beta + 396\alpha\beta^2 + 1122\beta^3$$

Substituting the values of X and T in (2,) we get

$$\begin{aligned} x(\alpha,\beta) &= 29\alpha^3 + 702\alpha^2\beta + 5742\alpha\beta^2 + 15444\beta^3 \\ y(\alpha,\beta) &= 39\alpha^3 + 957\alpha^2\beta + 7722\alpha\beta^2 + 21054\beta^3 \end{aligned} \quad (14)$$

Thus(14) and (12) represent non-zero distinct integral solutions of (1).

Properties:

- $29y(\alpha,\alpha+1)-39x(\alpha,\alpha+1)=750P_\alpha^5+8250CP_\alpha^6$
- $957x(\beta+1,\beta)-702y(\beta+1,\beta)+375CP_\beta^6+148500PP_\beta\equiv 0$
- $29y(t_{3,\alpha},t_{3,\alpha+2})-39x(t_{3,\alpha},t_{3,\alpha+2})=18000(t_{3,\alpha}*Pt_\alpha)+8250(\Pr_{\alpha+2})^3$
- $957x(\alpha,\alpha)-702y(\alpha,\alpha)+597CP_{5\alpha}^6\equiv 0$
- $29y(\alpha,\alpha)-39x(\alpha,\alpha)+69CP_{5\alpha}^6\equiv 0$

Illustration-2:

Write 25 as $25=(49+6\sqrt{66})(49-6\sqrt{66})$

Replacing the above process the non-zero distinct integral solutions of (1) are

$$x(\alpha,\beta)=85\alpha^3+2070\alpha^2\beta+16830\alpha\beta^2+45540\beta^3$$
$$y(\alpha,\beta)=115\alpha^3+2805\alpha^2\beta+22770\alpha\beta^2+61710\beta^3$$
$$z(\alpha,\beta)=\alpha^2-66\beta^2$$

Properties:

- $85y(\alpha,\alpha+1)-115x(\alpha,\alpha+1)=700P_\alpha^5-8250CP_\alpha^6$
- $2805x(1,2\beta+1)-2070y(1,2\beta+1)=74250Ky_\beta+148875$
- $85y(\alpha,1)-115x(\alpha,1)-375\Pr_\alpha\equiv 8250(\mathrm{mod}\,375)$
- $6171x(\beta+1,\beta)-4544y(\beta+1,\beta)+825CP_\beta^6+326700PP_\beta\equiv 0$

3. CONCLUSION:

In this paper, we have presented four different patterns of non-zero distinct integer solutions of the cube Diophantine equation given by $(3a+5)x^2-3ay^2=125z^3$. To conclude, one may search for other patterns of solutions and their corresponding properties.

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